Math Camp **Summer 2018**

Lecture 7: August

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In standard calculus courses we all encounter the notion of derivative and learn how to compute and apply them. However, now we will go a little bit deeper and will try to understand the theory behind calculus.

7.1 Derivative

Suppose f is a function defined on an interval I and let x_0 and x be points of I. The usual difference quotient is given by

$$\frac{f(x) - f(x_0)}{x - x_0}$$

, which is the slope of the chord determined by the point $(x, f(x))$ and $(x, f(x_0))$ or in other way we can interpret the quotient as the average rate of change of f on the interval with endpoints at x and x_0.

The usual definition of derivative is following

Definition 7.1 *Let f be defined on an interval I and let $x_0 \in I$. The derivative of f at x_0, denoted by $f'(x_0)$ is defined as*

$$f'(x_0) = \lim_{x \to x_0} [\frac{f(x) - f(x_0)}{x - x_0} \tag{7.1}$$

provided this limit exists.

If f is differentiable at all points $x \in I$, we say that f is differentiable on I.

Example 1 *Let $f(x) = x^2$ and let $x_0 \in \mathbb{R}$. If $x \in \mathbb{R}, x \neq x_0$, then*

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - {x_0}^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{x - x_0} = x + x_0$$

Since $x \neq x_0$, taking the limit we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

, giving as the derivative,

$$f'(x_0) = 2x_0$$

7.1.1 Differentiability and Continuity

A continuous function need not be differentiable (for example f(x) =—x—), however, the differentiable function is always continuous.

[0]All errors are my own.

Theorem 7.2 *Let f be defined in a neighborhood I of x_0. If f is differentiable at x_0, then f is continuous at x_0.*

We can use this theorem in two ways.I f we know that a function has a discontinuity at a point, then we know immediately that there is no derivative there.O n the other hand, if we have been able to determine by some means that a function is differentiable at a point then we know automatically that the function must also be continuous at that point.

7.1.2 Combinations of Differentiable Functions

Functions can be combined algebraically by multiplying by constants, by addition and subtraction, by multiplication, and by division.T o each of these there is a calculus rule for computing the derivative.

Theorem 7.3 *Let f and g be functions defined on an interval I, and assume both are differentiable at some point $x_0 \in I$. Then,*

1. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
2. $(kf)'(x_0) = kf'(x_0)$
3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
4. $(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$, *given that* $g(x_0) \neq 0$

The proof of (1) and (2) follow easily from the definition of the derivative and appropriate limit theorems and left as an homework. I will show the proof for part (3) during class.

Example 2
1. $f(x) = c$
2. $f(x) = ax + b$
3. $f(x) = x^n$
4. $f(x) = \frac{2x}{1-x^2}$

7.1.3 Chain Rule

With the differentiation formulas developed thus far, we can find derivative of functions f, for which $f(x)$ is finite sum of products or quotients. However, we have not learned to deal with the composite functions of the form $h \circ g$, which can be defined by the equation

$$f(x) = h(g(x))$$

, where h and g are functions such that the domain of h includes the range of g.

The *chain rule* tells us how to express the derivative of f in terms of the derivatives h' and g'.

Theorem 7.4 (Chain Rule) *Let f be the composition of two functions h and g, say $f = h \circ g$. Suppose that both derivatives $g'(x)$ and $h'(y)$ exists, where $y = g(x)$. Then the derivative $f'(x)$ also exists and is given by the formula*

$$f'(x) = h'(y) \cdot g'(x) \tag{7.2}$$

Instead of giving the proof we will solve some examples, so you will be familiar with the usual techniques.

Example 3 *1.* $f(x) = \sqrt{1+x^2}$

2. $f(x) = \frac{x}{sqrt4-x^2}$

3. $g(x) = f(x^2)$

4. $f(x) = [g(x)]^n$

7.1.4 Local Extrema

We have seen in previous note that a continuous function defined on a closed interval $[a, b]$ achieves an absolute maximum value and an absolute minimum value on the interval.So there must be points where the maximum and minimum are attained. But how do we go about finding such points? The idea is to look for the critical points (i.e., points where the derivative is zero).

Theorem 7.5 *Let f be defined on an interval I. If f has a local extremum at a point x_0 in the interior of I and f is differentiable at x_0, then $f'(x_0) = 0$.*

The proof will be provided during the class.

7.1.5 Mean Value Theorem

The question we are trying to answer in this section is how does information about the derivative provide us with information about the function? One of the keys to providing that information is the mean value theorem. Before formally defining and proofing Mean Value Theorem[1], we need to state one more theorem.

Theorem 7.6 (Rolle's Theorem) *Let f be a function which is continuous everywhere on a close interval $[a, b]$ and has a derivative at each point of the open interval (a, b). Also, assume that*

$$f(a) = f(b)$$

. Then there is at least one point c in the open interval (a, b) such that $f'(c) = 0$.

We will not proof this theorem, but the Figure 7.1 gives the geometric interpretation of Rolle's theorem and should be enough to understand the concept.

Now we use Rolle's theorem to prove the mean value theorem.

Theorem 7.7 (Mean Value Theorem For Derivatives) *Assume that f is continuous everywhere on a closed interval $[a, b]$ and has a derivative at each point of the open interval (a, b). Then there is at least one interior point c of (a, b) for which*

$$f(b) - f(a) = f'(c)(b - a) \tag{7.3}$$

The proof will be provided during the class.

[1] I decided to proof this theorem, since you have similar problems in your Econ 629 class.

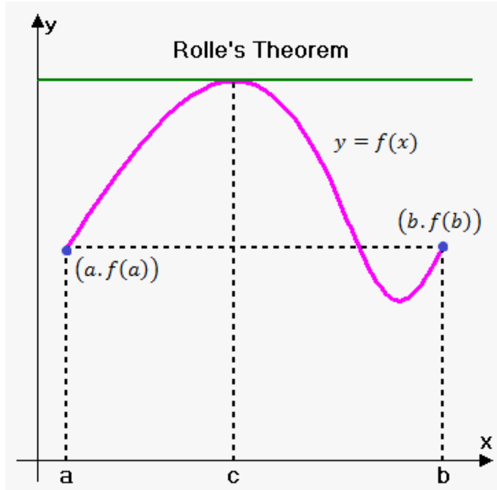

Figure 7.1: Geometric interpretation of Rolle's theorem. (Figure borrowed from Apostol, 1969.)

7.1.6 Application of the Mean Value Theorem

Now we use mean value theorem to deduce properties of a function from a knowledge of the sign of its derivative.

Theorem 7.8 *Let f be differentiable on an interval I.*

1. *If $f'(x) \geq 0$ for all $x \in I$, then f is nondecreasing on I.*
2. *If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.*
3. *If $f'(x) \leq 0$ for all $x \in I$, then f is nonincreasing on I.*
4. *If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.*
5. *If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*

We will prove (1) during the class, the rest is left as an homework.

7.1.7 Convexity and Derivative

The geometric properties we wish to capture when we say a function is convex or concave do not depend on differentiability properties. The condition is that the graph should lie below (or above) all its chords. The following definitions make this concept precise.

Recall that, we say that the function is convex if,

Definition 7.9 *If for all $x_1, x_2 \in I$ and $\alpha \in [0, 1]$ the inequality*

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2)$$

is satisfied.

If the reverse inequality is applied, we say that f is concave on I. If the inequalities are strict for all $\alpha \in (0, 1)$ we say f is strictly convex or strictly concave on I.

7.1.8 L'Hopital's Rule

Suppose that f and g are defined in a neighborhood of x_0 and that

$$\lim_{x \to x_0} f(x) = A \text{ and } \lim_{x \to x_0} g(x) = B.$$

According to our usual theory of limits, we then have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{A}{B},$$

unless $B = 0$. The question we want to answer is what happens if $B = 0$? Let us try to generalize from these two examples. Suppose f and g are differentiable in a neighborhood of $x = a$ and that $f(a) = g(a) = 0$. Consider the following calculations and what conditions on f and g are required to make them valid.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)} \xrightarrow{x \to a} \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

If these calculations are valid, they show that under these assumptions (f(a) = g(a) = 0 and both f(a) and g(a) exist) we should be able to claim that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Now we provide a theorem of the $\frac{0}{0}$ rule identical with our introductory remarks.

Theorem 7.10 *Suppose that the functions f and g are differentiable in a neighborhood N of $x = a$. Suppose*

1. $\lim_{x \to a} f(x) = 0$,
2. $\lim_{x \to a} g(x) = 0$,
3. *For every* $x \in N, g'(x) \neq 0$, *and*
4. $= \lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists*

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example 4 *Evaluate*

1. $\lim_{x \to 0} \ln(1 + x)/x$
2. $\lim_{x \to 0} (1 + x)^{2/x}$.
3. $\lim_{x \to 0} sinx/x$

7.2 References

References

[1] Abbott, S. *Understanding Analysis.* Springer-New York., 2001.

[2] Apostol, T.M.. *Calculus.* Blaisdell Pub. Co., 1969.

[3] Thomson, B.S., Brunckner, J.B, and Brunckner, A.M. *Elementary Real Analysis.* Prentice Hall (Pearson), 2001.

[4] Wade, W.R. *An Introduction to Analysis* Pearson Education, 2004.