Math Camp **Summer 2018**

Lecture 6: August

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6.1 Continuity

In this part of notes we deal with the concept of continuity, one of the fascinating ideas in all of mathematics. Before we give a technical definition of continuity, let briefly give the informal discussion fo the concept in order to understand the intuition of the continuity.

6.1.1 Informal Description

Suppose a function f has the value $f(p)$ at a certain point point p. Then we say f is continuous at p if at every nearby point x the function value $f(x)$ is close to $f(p)$. In other words, if we move x toward p, we want the corresponding function values $f(x)$ to become arbitrarily close to $f(p)$, regardless how x approaches to p. The intuition is we do not want sudden jumps in the values of a continuous function. Figure 6.1 shows the graph of the function $f(x) = x - [x]$. At each integer, you can observe a *jump discontinuity*. For example, $f(-2) = 0$, but as x approaches to 2 from the left $f(x)$ approaches to the value 1 . Thus we have a discontinuity at -2. However, $f(x)$ does approach $f(-2)$ if we let x approach to -2 from the right. In a case like this, the function called is called *continuous from the right at -2* and *discontinuous from the left at -2*. Note that continuity at a point requires both continuity from the left and from the right.

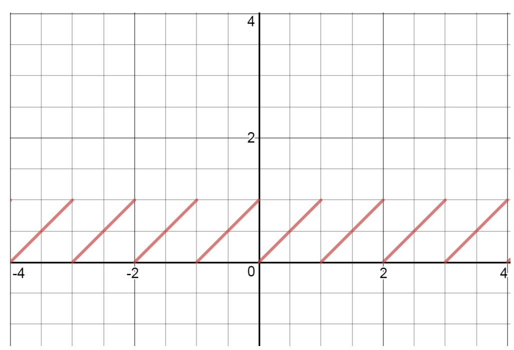

Figure 6.1: A jump discontinuity at each integer. (Figure borrowed from Apostol, 1969.)

6.1.2 Functional Limits

Now let understand what we mean by function approached to some value. Mathematically. the function limit is denoted

$$\lim_{x \to p} f(x) = L$$

[0]All errors are my own.

. It has the similar definition as a sequence limit. That is $f(x)$ gets arbitrary close to L as x is chosen closer and closer to p. Note that, in view of functional limits we are not interested on what happens when $x = p$. Recall that we define a limit point (accumulation point) $p \in E$, as a point with the property that every neighborhood $N(\epsilon, p)$ intersects E in some point other than p. Now we are ready to define the functional limit.

Definition 6.1 *Let $f : E$ rightarrow$\mathbb{R}$, and p be a limit function of the domain E. Then we write*

$$\lim_{x \to p} f(x) = L$$

, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that

$$|f(x) - L| < \epsilon$$

Note, that condition $0 < |x - c| < \delta$ is the same as $x \in N(\delta, x)$ or $x \in (p - \delta, p + \delta)$. Figure 6.2 shows a graphical interpretation of the definition. We now present some examples illustrating how to prove the

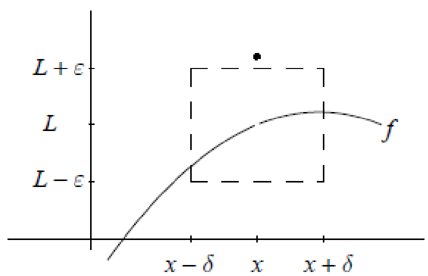

Figure 6.2: Graphical interpretation of the $\epsilon - \delta$ limit definition. (Figure borrowed from Thomson et.al, 2001.)

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existence of a limit directly from the definition.T hese are to be considered as exercises in understanding the definition.W e would rarely use the definition to compute a limit, and we hope seldom to use the definition to verify one; we will use the definition to develop a theory that will verify limits for us.

Example 1 *Prove that if*

1. $f(x) = 3x + 1$, *then* $\lim_{x \to 2} f(x) = 7$
2. $g(x) = x^2$, *then* $\lim_{x \to 2} g(x) = 2$

We proof only part 1.

Proof: Let $\epsilon > 0$, then the Definition 6.1 requires that we produce $\delta > 0$ so that $0 < |x - 2| < \delta$ gives as the conclusion $|f(x) - 7| < \epsilon$. Let directly apply the definition

$$|f(x) - 7| = |(3x + 1) - 7| = 3|x - 2|$$

. Thus, if we choose $\delta = \epsilon/3$ then $0 < |x - 2| < \delta$ implies

$$|f(x) - 7| < 3(\epsilon/3) = \epsilon$$

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6.1.3 Continuity at a Point

As long as we understand the functional limit, we can be ready to define the concept of continuity. We begin by defining continuity at a point, more specifically continuity at an interior point of the domain of a function f.

Definition 6.2 *A function $f : E \to \mathbb{R}$ is continuous at a point $p \in E$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - p| < \delta$ it follows that $|f(x) - f(p)| < \epsilon$.*

In other words the function f is continuous at point p if $\lim_{x\to p} f(x) = f(p)$. If f is continuous at every point in the domain E, then we say that f is continuous on E. Observe that a function f can fail to be continuous at p in three ways:

1. f is not defined at p.
2. $\lim_{x\to p} f(x)$ fails to exist.
3. f is defined at p and $\lim_{x\to p} f(x)$ exists, but $\lim_{x\to p} f(x) \neq f(p)$

Example 2 *Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Show that if $p \in (0, \infty)$ then f is continuous at p. Hint: Use Figure 6.3.*

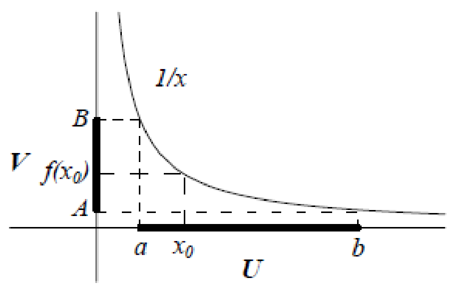

Figure 6.3: Graphical interpretation of the neighborhood definition of continuity for the function $f(x) = 1/x$. (Figure borrowed from Thomson et.al, 2001.)

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6.1.4 Properties of Continuous Functions

In this section we present some of the most basic of the properties of continuous functions.

Theorem 6.3 *Let $f, g : E \to \mathbb{R}$ and let $c \in \mathbb{R}$. Sippose f and g are continuous at $p \in \mathbb{E}$. Then $cf, f + g\ fg$ are continuous at p. Moreover, if $g(p) \neq 0$, then f/g is continuous at p.*

Example 3 *Show that every polynomial is continuous on $\mathbb{R}$*

Theorem 6.4 *Let $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and suppose that $f(A) \subset \mathbb{R}$. If F is continuous at a point $p \in A$ and g is continuous at the point $y = f(p) \in B$. Then the composition function*

$$g \circ f : A \to \mathbb{R}$$

is continuous at p

Example 4 *Suppose for $x \in (0, \infty)$ we have the following discontinuous function*

$$f(x) = \begin{cases} 2x, & \text{if } x < 3 \\ 2x + 4, & \text{if } x \in [3, 6) \\ 2x + 6, & \text{if } x \geq 6 \end{cases}$$

Can you define the new function which "fix" the discontinuous function and makes it continuous. Hint: Use 6.4 for the intuition.

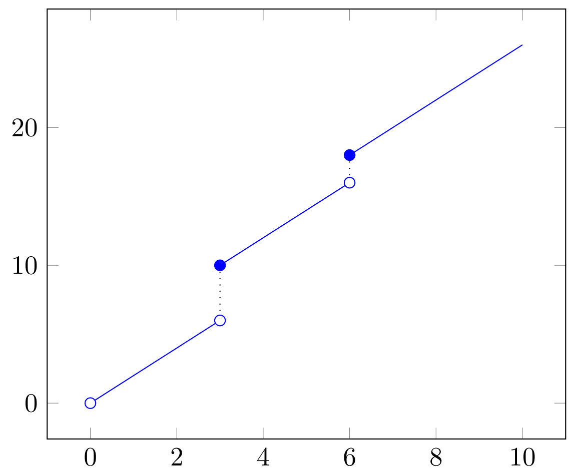

Figure 6.4: Plot of the discontinuous function $f(x)$

6.2 Continuous Functions on Compact Set

Definition 6.5 *Given a function $f : A \to \mathbb{R}$. Then f is bounded if $\exists\, M \in \mathbb{R}_+$, such that $|f(x)| \leq M, \forall x \in A$.*

Example 5 *Let $f :\to \mathbb{R}$, where $f(x) = \frac{1}{1+x^2}$. Show that when*

1. *$A := [0, \infty)$, f is unbounded*
2. *$A := [0.5, \infty)$, f is bounded*

Recall that the set is compact, if it is closed and bounded.

Theorem 6.6 *Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

An extremely important corollary is obtained by combining this result with the observation that compact sets are bounded and contain theirs supremums and infimums.

Theorem 6.7 (Extreme Value Theorem) . *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value.*

In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in K$

6.3 References

References

[1] Abbott, S. *Understanding Analysis.* Springer-New York., 2001.

[2] Apostol, T.M.. *Calculus.* Blaisdell Pub. Co., 1969.

[3] Thomson, B.S., Brunckner, J.B, and Brunckner, A.M. *Elementary Real Analysis.* Prentice Hall (Pearson), 2001.

[4] Wade, W.R. *An Introduction to Analysis* Pearson Education, 2004.