Math Camp	**Summer 2018**

Lecture 5: August

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Modern set theory has had been introduced to us by the young mathematician Georg Cantor in 1870. The most talented analysts of that time began applying his ideas to the theory of functions, and by now this material is essential to an understanding of the subject. In this note we introduce the most basic material. In the words of the mathematician David Hilbert, "No one shall expel us from the paradise that Cantor has created for us."

The ideas in this section are all very geometric.Try to draw mental images that depict all of these ideas to get a feel for the definitions.The definitions themselves should be remembered but may prove hard to remember without some associated picture.

5.1 Important Points

In our studies of analysis we shall often need to have a language that describes sets of points and the points that belong to them. Every point inside an open interval $I = (a, b)$ has the feature that there is a smaller open interval centered at that point that is also inside I. Thus if $x \in (a, b)$ then for any positive number c that is small enough

$$(x - c, x + c) \subset (a, b)$$

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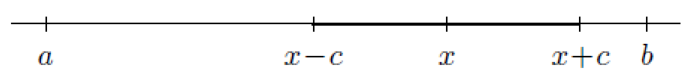

Figure 5.1: Every point in (a, b) is an interior point. (Figure borrowed from [Thomson et. al 2001].)

It is easy to show,let c be any positive number that is smaller than the shortest distance from x to either a or b.Then $(x - c, x + c) \subset (a, b)$.

In literature you will see following language. An open interval that contains a point x is said to be a *neighborhood* of x. Thus each point in (a, b) possesses a neighborhood, indeed many neighborhoods, that lie entirely inside the set I.On occasion the point x itself is excluded from the neighborhood: We say that an interval (c, d) is a neighborhood of x if x belongs to the interval and we say that the set $(c, d) \setminus \{x\}$ is a deleted neighborhood. This is just the interval with the point x removed.

Definition 5.1 *Let E be a set of real numbers. Any point x that belongs to E is said to be an interior point of E provided that some interval*

$$(x - c, x + c) \subset E$$

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[0]All errors are my own.

Most sets that we consider will have infinitely many points.Certainly any interval (a, b) or $[a, b]$ has infinitely many points.The set $\mathbb{N}$ of natural numbers also has infinitely many points, but as we look closely at any one of these points we see that each point is all alone, at a certain distance away from every other point in the set. We call these points *isolated points* of the set.

Definition 5.2 *Let E be a set of real numbers. Any point x that belongs to E is said to be an isolated point of E provided that for some interval $(xc, x + c)$*

$$(x - c, x + c) \cap E = x$$

.

While the isolated points are of interest on occasion, more than likely we would be interested in points that are not isolated. These points have the property that every containing interval contains many points of the set.

Definition 5.3 *Let E be a set of real numbers. Any point x (not necessarily in E) is said to be an accumulation point (limit point) of E provided that for every $c > 0$ the intersection $(xc, x + c) \cap E$ contains infinitely many points.*

The intervals (a, b) and $[a, b]$ have what appears to be an edge.The points a and b mark the boundaries between the inside of the set (i.e., the interior points) and the outside of the set.

Definition 5.4 *Let E be a set of real numbers. Any point x (not necessarily in E) is said to be a boundary point of E provided that every interval $(x - c, x + c)$ contains at least one point of E and also at least one point that does not belong to E.*

Example 1 *Find the interior,isolated,accumulation points for the following sets*

1. *Open interval (a, b).*
2. *Closed interval $[a, b]$.*
3. *The set of natural numbers $\mathbb{N}$.*
4. *Set of rational numbers $\mathbb{Q}$*

5.2 Elementary Topology of Sets

Topology is a branch of mathematics and is fundamental to an understanding of many areas of mathematics. We describer open and closed sets on the real line and describe precisely what they are.

5.2.1 Closed Sets

Definition 5.5 *Let E be a set of real numbers. The set E is said to be closed provided that every limit point of E belongs to the set E.*

Thus a set E is not closed if there is some accumulation point of E that does not belong to E. That is if we want to make the set closed, we need to add "the missing " points to it. Note, a set with no accumulation points would have to be closed since there is no point that needs to be checked.

Definition 5.6 *Let E be any set of real numbers and let E' denote the set of all accumulation points of E. Then the set*

$$\bar{E} = E \cup E'$$

is called the closure of the set E.

Example 2 *Find which of this sets are closed and determine their closure.*

1. *Open interval (a,b)*
2. *Closed interval [a,b]*
3. $\mathbb{N}$
4. *Rational numbers* $\mathbb{Q}$

5.3 Open Sets

Definition 5.7 *Let E be a set of real numbers. Then E is said to be open if every point of E is also an interior point of E.*

In other words if for every point $x \in E$, there is an neighborhood $N(x, \epsilon)$, such that $N(x, \epsilon) \in E$.It is immediate that an open set cannot contain any of its boundary points. Thus if a set is not open it is because it contains points which are not interior. If we get rid of them than we will end up with a smaller set composed entirely of the interior points.

Definition 5.8 *Let E be any set of real numbers. Then the set*

$$int(E)$$

denotes the set of all interior points of E and is called the interior of the set E.

Example 3 *Find which of this sets are open and determine their interior.*

1. *Open interval (a,b)*
2. *Closed interval [a,b]*
3. $\mathbb{N}$
4. *Rational numbers* $\mathbb{Q}$

Note that if the set is open it **does not** mean it is closed. Can you give a simple example?

5.3.1 Topology

The study of open and closed sets in any space is called topology.Our goal now is to find relations between these ideas and examine the properties of these sets.

Theorem 5.9 *Let A be a set of real numbers and $B = R \setminus A$ its complement. Then A is open if and only if B is closed.*

Now we list some propertied of open and closed sets, without giving proofs.

Theorem 5.10 *Open sets of real numbers have the following properties:*

1. *The sets $\emptyset$ and $\mathbb{R}$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

Theorem 5.11 *Closed sets of real numbers have the following properties:*

1. *The sets $\emptyset$ and $\mathbb{R}$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

5.3.2 Compactness

In analysis we frequently encounter the problem of arguing from a set of local assumptions to a global conclusion. Compactness is one of the fundamental notions which helps as to generalize our assumptions.

Definition 5.12 *A set E is compact if every sequence in E has a subsequence that converges to a limit that is also in E*

One of the important theorems that you will use frequently is **Heine-Borel Theorem**.

Theorem 5.13 *A set E is compact if and only if it is closed and bounded.*

There is one more theorem of compactness based on open covers, but we omit that in our note.

Example 4 *Decide which of the following sets are compact*

1. $\mathbb{Q}$
2. *Closed interval* $[a, b]$

5.4 References

References

[1] Thomson, B.S., Brunckner, J.B, and Brunckner, A.M.. *Elementary Real Analysis.* Prentice Hall (Pearson), 2001.

[2] Wade, W.R. *An Introduction to Analysis* Pearson Education, 2004.