Math Camp	**Summer 2018**

Lecture 4: August

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Now we are ready to introduce and define the notion of sequence.

4.1 Sequences

A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a **finite list is not called a sequence; a sequence must continue without interruption**. For a more formal definition notice that the natural numbers are playing a key role here.Every item in the sequence (the list) can be labeled by its position; label the first item with a "1," the second with a "2," and so on.Seen this way a sequence is merely then a function mapping the natural numbers $\mathcal{N}$ into some set. We state this as a definition. Since this chapter is exclusively about sequences of real numbers, the definition considers just this situation.

Definition 4.1 *A sequence of real numbers is a function*

$$f : \mathcal{N} \to \mathcal{R}$$

Thus we can frite sequence as

$$f(1), f(2), \dots, f(n), \dots$$

. The function values $f(1), f(2), \dots$ are called the *terms* of the sequence. Now we give several famous examples of sequences.

Example 1

- ***Arithmetic Progression****: The sequence*

$$c, c+d, c+2d, \dots, c+(n-1)d$$

, where number d is the common difference or as a formula

$$x_n = c + (n-1)d$$

- ***Geometric Progressions****: The sequence*

$$c, cr, cr^2, cr^3, \dots, cr^n, \dots$$

. The number r is called the common ratio or as a formula

$$x_n = cr^{n-1}$$

A sequence f whose terms are $x_n := f(n)$ will be denoted by $x_1, x_2, \dots$ or $\{x_n\}_{n\in\mathcal{N}}$. Thus $1, 1/2, 1/4, \dots$ represents the sequence $\{1/2^{n-1}\}_{n\in\mathcal{N}}$ and $-1, 1, -1, 1, \dots$ represents the sequence $\{(-1)^n\}$, finally $1, 2, 3, 4, \dots$ represents the sequence $\{n\}$.

[0]All errors are my own.

Definition 4.2 *A nonempty set S of real numbers is said to be countable if there is a sequence of real numbers whose range is the set S.*

In the language of this definition then we can see that (1) any finite set is countable, (2) the natural numbers and the integers are countable, (3) the rational numbers are countable, and (4) no interval of real numbers is countable.By convention we also say that the empty set $\emptyset$ is countable.

4.1.1 Convergence

From elementary calculus , we know that the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

is getting closer and closer to the number 0. In other words the sequence converges to 0 or that the limit of the sequence is the number 0. The limit concept is one of the fundamental building blocks of analysis. How to define *getting closer* idea?

Definition 4.3 *A sequence of real numbers $\{x_n\}$ is said to converge to a real number $a \in \mathcal{R}$ is and only if for every $\epsilon > 0$ there is an $N \in \mathcal{N}$ such that*

$$n \geq N \text{ implies } |x_n - a| < \epsilon$$

The following phrases have the same meaning

- $\{x_n\}$ converges to a.
- $a = \lim_{n\to\infty} x_n$.
- $x_n \to L$ as $n \to \infty$

When $x_n \to a$ as $n \to \infty$, think of x_n as a sequence of approximations to a, and ϵ as an upper bound for the error of these approximations. So we choose number N such that the error is less than ϵ when $n \geq N$. Generally smaller ϵ implies larger N.

One thing you can notice from the definition is that x_n converges to a if and only if $|x_n - a| \to 0$ as $n \to \infty$. In particular, $x_n \to 0$ if and only if $|x_n| \to 0$ as $n \to \infty$.

Figure 4.1: Example of converged sequence. (Figure borrowed from [wade2004].)

Example 2 1. *Prove that $1/n \to 0$ as $n \to \infty$.*

2. *Does $\{(-1)^n\}$ converge or diverge.*

In order to make you comfortable with proofing things, we will go head and proof the following lemma.

Lemma 4.4 *A sequence have at most one limit.*

Proof: We proof by contradiction. Suppose that x_n converges to both a and b. From definition of convergence we know that $\exists N_1$ and $N_2 : n \geq N_1 \Rightarrow |x_n - a| < \epsilon/2$ and $n \geq N_2 \Rightarrow |x_n - b| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$, By the choice of N_1 and $N_2, n \geq N$ implies both $|x_n - a| < \epsilon/2$ and $|x_n - b| < \epsilon/2$. Therefor from triangle inequality

$$|a - b| \leq |a - x_n| + |x_n - b| < \epsilon \Rightarrow |a - b| < \epsilon \ \ \forall \epsilon > 0$$

. Thus $a < b + \epsilon \Rightarrow a \leq b$ and $b < a + \epsilon \Rightarrow b \leq a \Rightarrow a = b$ ■

We say that the sequence $\{x_n\}$ is **bounded above** if $\exists x \in \mathcal{R}$ and a real number M, such that $\forall n \in \mathcal{N}, x_n \leq M$. Similarly it is *bounded below* if and only if $\exists m \in \mathcal{R} : X_n \geq m \forall n \in \mathcal{N}$. The good question is, is there a relationship between convergent sequences and bounded sequences? The answer is yes.

Theorem 4.5 *Every convergent sequence is bounded,*

Proof: Take $\epsilon > 1$, since the sequence is convergent $\exists N \in \mathcal{N} : n \geq N \Rightarrow |x_n - a| \leq 1$. Hence by triangular inequality $|x_n| \leq 1 + |a| \ \forall n \geq N$. On the other hand for $1 \leq n \leq N$, we have

$$|x_n| \leq M := \max\{|x_1|, |x_2|, \ldots, |x_N|\}$$

. Thus $\{x_n\}$ bounded by$|x_n| \leq \max\{M, 1 + |a|\}$. ■ There are two famous theorems that help as to proof things using sequences.

Theorem 4.6 (Squeeze theorem) *Let $\{x_n\}, \{y_n\}$ and $\{w_n\}$ be real sequences.*

1. *If $x_n \to a$ and $y_n \to a \Rightarrow \exists N_0 \in \mathcal{N} : x_n \leq w_n \leq y_m \ \ \forall n \geq N_0$ then $w_n \to a$ as $n \to \infty$.*
2. *If $x = \lim_{n \to \infty} x_n$. Then*
$$a \leq x_k \leq b \ \forall k = 1, 2, \cdots \Rightarrow a \leq x \leq b$$
.

Sometimes we want "correct" a sequence, for example to make it converge faster. To do that we introduce the notion of **sub-sequence**.

Definition 4.7 *A sub-sequence of a sequence $\{x_n\}$, is a sequence of the form $\{x_{n_k}\}$, where $n_k \in \mathcal{N}$ and $n_1 < n_2 < \ldots$*

Thus a sub-sequence $x_{n_1}, x_{n_2}, \ldots$ of $x_1, x_2, \ldots$ is obtained by deleting from $x_1, x_2, \ldots$ all x_n's except those such that $n = n_k$ for some k.

Example 3 $1, 1, 1, \ldots$ *is a sub-sequence of $(-1)^n$ obtained by deleting every other term ($n_k = 2k$)*

If a sub-sequence converges, the limit called a *limit point* of $\{x_n\}$.

4.1.2 Monotone Sequences

The interesting thing you may notice is that although the sequence $\{-1\}^n$ does not converge, however it has convergent subseqence (recall Example 2). This is not a coincidence, we will see that every bounded sequence has a convergent subsequence. Lat start by defining monotone sequences.

Definition 4.8 *Let $\{x_n\}$ be a sequence of real numbers*

1. *$\{x_n\}$ is said to be increasing(strictly increasing) if and only if $x_1 \leq x_2 \leq x_2 \leq \ldots$ (respectively, $x_1 < x_2 < x_3 < \ldots$)*
2. *$\{x_n\}$ is said to be decreasing(strictly decreasing) if and only if $x_1 \geq x_2 \geq x_2 \geq \ldots$ (respectively, $x_1 > x_2 > x_3 > \ldots$)*
3. *$\{x_n\}$ is said to be monotone if and only if it is either increasing or decreasing.*

In Theorem (4.5) we show that every convergent sequence is bounded, now we establish the converse result for the monotone sequences.

Theorem 4.9 *If $\{x_n\}$ is monotone and bounded, then it converges. In other words $\{x_n\}$ has a finite limit.*

Let skip the proof of this theorem and do some examples.

Example 4 1. *If $a > 0$, then $a^{1/n} \to 1$ as $n \to \infty$*

2. *If $|a| < 1$, then $a^n \to 0$ as $n \to \infty$*

We will solve only second part.
Proof: The first thing you should notice is that it is suffices to prove that $|a|^n$ as $n \to \infty$. (Can you bring legitimate argument for this?). Note that $|a|^n$ is monotone decreasing, since $|a| < 1 \implies |a|^{n+1} < a^n$ and $|a|^n$ is bounded below by 0 (why?). Thus from Theorem 4.9 $L := \lim_{n\to 0} \infty$ exists.
Now we prove that $L = 0$. Suppose by contradiction not, i.e $L \neq 0$. Since $|a|^{n+1} = |a| \cdot |a|^n$, and taking limits from both sides when $n \to \infty$, we see that $L = |a| \cdot L$. However since L is not zero, therefore $|a| = 1$, but this is a contradiction that $|a| < 1$. ■

4.1.3 Limits Supremum and Infimum

In the future we will deal with the situations, when we need generalization of limits.

Definition 4.10 *Let $\{x_n\}$ be a sequence (real). Then the limit supremum of $\{x_n\}$ is the extended real number*

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{k\geq n} x_k)$$

and the limit infimum

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{k\geq n} x_k)$$

In interpreting this definition note that, by our usual rules on infs and sups, the values ∞ and $\infty$are allowed.

$$\limsup_{n\to\infty} x_n = \infty \iff \{x_n\} \text{ has no upper bounds}$$

Similarly for the infimum. Let give a closer look to Definition (4.10). Consider the sequences

$$s_n = \sup_{k\geq n} x_k := \sup\{x_k : k \geq n\} \text{ and } t_n = \inf_{k\geq n} x_k := \inf\{x_k : k \geq n\}$$

If you stare on this expressions a little bit, you may notice that s_n is decreasing and t_n is increasing. Thus if x_n is bounded then the usual limit always exist.

Example 5 *Find* $\limsup_{n\to\infty} x_n$ *and* $\liminf_{n\to\infty} x_n$

1. $x_n = (-1)^n$
2. $x_n = 1 + 1/n$

4.2 References

References

[1] Thomson, B.S., Brunckner, J.B, and Brunckner, A.M.. *Elementary Real Analysis*. Prentice Hall (Pearson), 2001.

[2] Wade, W.R. *An Introduction to Analysis* Pearson Education, 2004.