Math Camp	**Summer 2018**

Lecture 3: August

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To do real analysis we should know exactly what the real numbers are. Here we give a loose exposition to real numbers.

3.1 Properties of the Real Numbers

We start with the **natural numbers**.These are the counting numbers

$$1, 2, 3 \ldots$$

.The symbol $\mathcal{N}$ is used to indicate this collection. Thus $n \in \mathcal{N}$ means that n is a natural number, one of these numbers $1, 2, 3, 4, \ldots$.

Usually the natural numbers prove to be rather limited in representing problems that arise in applications of mathematics to the real world.Thus they are enlarged by adjoining the negative integers and zero. Thus the collection

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

is denoted $\mathcal{Z}$ and called **the integers**.

At some point the problem of the failure of division in the sets $\mathcal{N}$ and $\mathcal{Z}$ becomes acute and the student must progress to an understanding of fractions. This larger number system is denoted $\mathcal{Q}$, where the symbol chosen is meant to suggest quotients, which is after all what fractions are. The collection of all numbers of the form

$$\frac{m}{n}$$

where $m \in \mathcal{Z}$ and $n \in \mathcal{N}$ is called the set of **rational numbers** and is denoted $\mathcal{Q}$.

We often refer to the real number system as the real line and think about it as a geometrical object. Not all real numbers are rational. For example $\sqrt{2}$. The set of real numbers that are not rational are called **irrational**.

Example 1 *Show that $\sqrt{2}$ is not a rational number.*

Note: The real number system is an ordered field.

3.1.1 Bounds

Let E be some set of real numbers. There may or may not be a number M that is bigger than every number in the set E.If there is, we say that M is an upper bound for the set. If there is no upper bound, then the set is said to be *unbounded above* or to have no upper bound. This is a simple but critical idea, which helps understand the real numbers.

[0]All errors are my own.

Definition 3.1 *Let E be a set of real numbers. A number M is said to be an upper bound for E if $x \leq M$ for all $x \in E$.*

Similarly

Definition 3.2 *Let E be a set of real numbers. A number M is said to be a lower bound for E if $x \geq M$ for all $x \in E$.*

A set that has an upper bound and a lower bound is called *bounded.* A set can have many upper bounds. Indeed every number is an upper bound for the empty set $\emptyset$.A set may have no upper bounds. We can use the phrase E is unbounded above if there are no upper bounds.

Definition 3.3 *Let E be a set of real numbers. f there is a number M that belongs to E and is larger than every other member of E, then M is called the maximum of the set E and we write $M = \max E$.*

Definition 3.4 *Let E be a set of real numbers. If there is a number m that belongs to E and is smaller than every other member of E, then m is called the minimum of the set E and we write $m = \min E$.*

Example 2 *What are the max, min, upper and lower bounds of the following sets*

1. $[0, 1] = \{x|0 \leq x \leq 1\}$
2. $(0, 1) = \{x|0 < x < 1\}$
3. *The set $\mathcal{N}$ of natural numbers.*

3.1.2 Sups and Infs

If E has a maximum, say M, then that maximum could be described by the statement M is the least of all the upper bounds of E, that is to say, M is the minimum of all the upper bounds. It is possible for a set to have no maximum and yet be bounded above.

Example 3 *The open interval $(0, 1)$ has no maximum, but many upper bounds. Certainly 2 is an upper bound and so is 1. The least of all the upper bounds is the number 1. Note that 1 cannot be described as a maximum because it fails to be in the set.*

Definition 3.5 *(Least Upper Bound/Supremum) Let E be a set of real numbers that is bounded above and nonempty.If M is the least of all the upper bounds, then M is said to be the least upper bound of E or the supremum of E and we write $M = \sup E$.*

The similar definition can be stated for the Greatest Lower Bound(Infimum).

Definition 3.6 *Let E be a set of real numbers that is bounded below and nonempty.If m is the greatest of all the lower bounds of E, then m is said to be the greatest lower bound of E or the infimum of E and we write $M = \inf E$.*

We write:

1. If E is unbounded above, then $\sup E = \infty$
2. If E is unbounded below, then $\inf E = \infty$

Any example of a nonempty set that you are able to visualize that has an upper bound will also have a least upper bound. The good question is, does this always true. It turns out that the answer of the question is yes. This property is known as **The axiom of Completeness**.

Theorem 3.7 *A nonempty set of real numbers that is bounded above has a least upper bound (i.e., if E is nonempty and bounded above, then $\sup E$ exists and is a real number).*

At this stage, I am not going to provide proof for this theorem.

3.1.3 Advanced Properties

In this part we give some advanced properties of $\mathcal{R}$. First we proof one of the important properties, which called **Archimedean Property**.

Theorem 3.8 *The set of natural numbers $\mathcal{N}$ has no ujpper bound.*

Proof: We proof by contradiction. Suppose by contradiction the set $\mathcal{N}$ does have an upper bound, then by the Axiom of Completeness it must have a least upper bound. Denote it by $x = sup\mathcal{N}$. Then $n \leq x$ for all integers n, however $n \leq x - 1$ cannot be true for all integers n. Therefore, we can choose some integer m with $m > x - 1$. Then $m + 1$ is also an integer and $m + 1 > x$. But that cannot be true since x is the supremum. Thus, contradiction follows. ∎

Some consequences of Archimedean Propery are:

- Given any positive number y, no matter how large, and any positive number x, no matter how small , one can find $n \in \mathcal{N}$, such that $nx > y$
- Given any positive number x, no matter how small, one can always find $n \in \mathcal{N}$, such that $1/n < x$

Example 4 *Let x be any real number. Show that there is an integer $m \in \mathcal{Z}$ so that*

$$m \leq x < m + 1$$

. Show that m is unique.

There is an important relationship holding between the set of rational numbers $\mathcal{Q}$ and the larger set of real numbers $\mathcal{R}$.The rational numbers are dense. They make an appearance in every interval; there are no gaps, no intervals that miss having rational numbers.

Definition 3.9 *A set E of real numbers is said to be dense if every interval (a, b) contains a point of E.*

Theorem 3.10 *The set $\mathcal{Q}$ of rational numbers is dense.*

The proof of the theorem heavily relies on the archimedean property. We omit the proof of this theorem.

3.1.4 The Metric Structure of $\mathcal{R}$

In practice, there are cases when we need to describe distances between points. These are the metric properties of the reals, to borrow a term from the Greek for measure (metron).

As usual, the distance between a point x and another point y is either xy or yx depending on which is positive. Thus the distance between 3 and 4 is 7. To describe this in general requires the absolute value function which simply makes a choice between positive and negative.

Definition 3.11 *For any real number x write*

$$|x| = x \textit{ if } x \geq 0$$

and

$$|x| = -x \textit{ if } x < 0$$

Theorem 3.12 *The absolute value function has the following properties:*

1. *For any $xin\mathcal{R}$, $-|x| \leq x \leq |x|$*
2. *For any $x, y \in \mathcal{R}$, $|xy| = |x||y|$*
3. *For any $x, y \in \mathcal{R}$, $|x + y| \leq |x| + |y|$*
4. *For any $x, y \in \mathcal{R}$, $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$*

Example 5 *Prove (2) and (4)*

Using the absolute value function we can define the distance function or metric.

Definition 3.13 *The distance between two real numbers x and y is*

$$d(x, y) = |xy|$$

Theorem 3.14 *The distance has the following properties*

1. *$|xy| \geq 0$ and $|xy| = 0$ iff $a = 0$*
2. *$|xy| = |y - x|$*
3. *$|x - y| \leq |x - z| + |z - y|$*

Example 6 *Prove (3)*

Example 7 *Show that*

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

3.2 References

References

[1] Thomson, B.S., Brunckner, J.B, and Brunckner, A.M.. *Elementary Real Analysis*. Prentice Hall (Pearson), 2001.

[2] Wade, W.R. *An Introduction to Analysis* Pearson Education, 2004.