Math Camp | **Summer 2018**

Lecture 1: August

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1.1 Strategies for proofs

In this section, we study the strategies to build proofs. Not all mathematics involves proofs. We learn arithmetic and introduction to calculus before we learn how to prove. The first use of proofs is attributed to Thales of Miletus from Alexandria. Euclid used an axiomatic system, which is needed for proofs. Statements that are accepted as true are called axioms. The question is, what is it that we need to prove in mathematics?

We prove statements which are usually called theorems, propositions, lemmas, corollaries, and exercises. Theorems tend to be important results; propositions are usually less important than theorems; lemmas are statements that are used in the proofs of other results; corollaries are statements that follow easily from other results.

Before discussing the ways to form the proofs, we fill a missing final ingredient in the formulation of theorems and proofs.

1.1.1 Quantifiers and Logical Operators

Assume we wish to prove mathematical expression "$x^3 > 27$". This expression as given is not precise since it does not state which possible values of x which are under consideration. A more useful statement would be "$x^3 > 27$, for all real numbers $x > 3$". The phrase for all real numbers $x > 3$ is an example of a quantifier.

Let give a closer look at the two main types of quantifiers. Let $P(x)$ be a statements and U some collection of possible values of x. A **universal quantifier(operator)** applied to $P(x)$ is a statement denoted ($\forall$ in $U)P(x)$, which is true if $P(x)$ is true for all possible values of x in U. The two possible ways to write ($\forall$ in $U)P(x)$ in English are:

- For all values of x in U, the statement $P(x)$ is true;
- For each x in U, the statement $P(x)$ is true;

The next type of quantifier is **existential quantifier**. An **existential quantifier** applied to $P(x)$ is a statement, denoted by ($\exists$ in $U)P(x)$, which is true if $P(x)$ is true for at least one value of x in U. The possible ways to write an existential quantifier in English are:

- For some value of x in U, we have $P(x)$ is true;
- There exists some x in U such that $P(x)$ holds;

Try to interpret the examples below. We will look on them closer during the class. Suppose, for example, that the possible values of x are all AGEC Ph.D students, the possible values of y are all types of classes offered by Texas A&M and $T(x, y)$ = person x takes class y.

[0] All errors are my own.

Example 1 *Interpret* $(\forall x)(\forall y)T(x, y)$

1. $(\forall y)(\forall x)T(x, y)$
2. $(\exists y)(\forall x)T(x, y)$
3. $(\exists x)(\exists y)T(x, y)$
4. *Negate* $(\forall x \in U)P(x)$
5. *Negate Example* 1.1*(a-d)*

The other basic logical operators are:

Operator	**Notation**	**Statement is true if:**
Negation	$\neg p$	p is not true
and	$p \wedge q$	p and q are both true
or	$p \vee q$	either p or q is true
there exists	$\exists x$ s.t $p(x)$	there exists x for which $p(x)$ is true
there exists a unique	$\exists! x$ s.t $p(x)$	there exists a unique x for which $p(x)$ is true
for all	$\forall x, p(x)$	for each x, $p(x)$ is true
implies	$p \Rightarrow q$	$(\neg p) \vee q$ is true
if and only if	$p \Leftrightarrow q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$ is true.

Figure 1.1: Logical operators. Adapted from "Unit 1: math camp" by Rodrigo A.Velez, Department of Economics, Texas A&M University

1.1.2 Direct Proofs

The statement of almost every theorem can be viewed in the form $P \rightarrow Q$, or some combination of such statements. For example,in Theorem 1.1, three parts are in the form $P \rightarrow Q$. We will proof it in class.

Theorem 1.1 *Let n and m be integers.*

(i) If n and m are both even, then $n + m$ is even.

(ii) If n and m are both odd, then $n + m$ is even.

(iii) If n is even and m is odd, then $n + n$ is odd.

The type of proof, which starts by assuming P is true, and produce a series of steps, each one following from the previous ones, and which eventually leads to Q, called a **direct proof**.

A direct proof presents a chain of consequences added to initial statement P in such a way that at each step a proposition that is true is added, reaching a final statement Q.

Example 2 *Let a, b and c be integers. If $a|b$*[1] *and $b|c$, then $a|c$.*

[1] $a|b$ means a divides b, that is there is some integer q such that $aq = b$

Example 3 *Any integer divides zero.*

Example 4 *Let n be an integer. Show that if n is even, then $3n$ is even.*

The usual direct proofs look something like "Suppose P. ... (argumentation) ... Then Q."

1.1.3 Proofs by Contrapositive and Contradiction

In this section we discuss two strategies for proving statements of the form $P \to Q$. Both strategies rely on the ideas developed in Unit 1. Recall that the contrapositive of $P \to Q$ (*Problem 6 in Unit 1*), namely $(\neg Q \to \neg P)$ is equivalent to $(P \to Q)$. Thus, in order to prove $P \to Q$, we could start by assuming that Q is false, and then by step-by-step argument go from $\neg Q$ to $\neg P$. A proof of this sort is called **proof by contrapositive**.

Example 5 *Let n be an integer. If n^2 is even, then is n is even.*

Another method of proof for theorems with statements of the form $P \to Q$ is **proof by contradiction**. In *Lecture 0:Problem 7*, we showed that $\neg(P \to Q) \iff (P \wedge \neg Q)$. That is, if we prove that $P \wedge \neg Q$ is false; we can conclude that $\neg(P \to Q)$ is false, and hence $\neg(\neg(P \to Q))$ is true. It then follows, using the Double Negation equivalence (*Lecture 0:Problem 6*), that $P \to Q$ must be true.

Example 6 *The number $\sqrt{2}$ is irrational*

Example 7 *The only consecutive non-negative integers a,b and c that satisfy $a^2 + b^2 = C^2$ are 3,4 and 5.*

1.1.4 Case, and If and Only If

The other commonly used method to prove a statement $P \to Q$ is by breaking up the proof into a number of cases. In *Lecture 0: Problem 3*, we showed that $(A \vee B) \to Q$ is equivalent to $(A \to Q) \wedge (B \to Q)$. Hence in order to prove that a statement of the form $A \vee B \to Q$ is true, we prove that the both statements $(A \to Q)$ and $(B \to Q)$ are true.

Example 8 *Let n be an integer. Then $n^2 + n$ is even.*

Next common logical form of the statement of a theorem is $P \leftrightarrow Q$. We refer such theorems as "if and only if" theorems. To prove such statements we use the result from the *Lecture 0, problem 4*. Thus, to prove a single statement of the form $P \leftrightarrow Q$, it will suffice to prove the two statements $P \to Q$ and $Q \to P$.

Example 9 *Let m and n be integers. Then mn is odd if and only if both m and n are odd.*

1.1.5 Mathematical Induction

Mathematical induction is a very useful method of proving certain types of statements that involve natural numbers ($\mathbb{N} = \{1, 2, 3, 4 \ldots\}$). More precisely, mathematical induction is a method that can be used to prove statements of the form $(\forall n \in \mathbb{N})(P(n))$, where $P(n)$ is some statements involving n. The way to construct proof by induction is following:

Theorem 1.2 *Let $P(n)$ is a statements involving n and $\forall n \in \mathbb{N}$*

1. *Provide a proof that $P(1)$ is true*
2. *Provide a proof of the statement, if $P(n)$ is true than $P(n+1)$ is true*
3. *Then $P(n)$ is true $\forall n \in \mathbb{N}$*

For the proof of the Theorem 1.2 look on Theorem 1, page 6 of *"Unit 1: math camp lecture notes" by Dr.Rodrigo A.Velez "*

Example 10 *Show that $8^n - 3^n$ is divisible by 5 $\forall n \in \mathbb{N}$*

Example 11 *Let $n \in \mathbb{N}$. Then*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Example 12 *Let $n \in \mathbb{N}$. Then*

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 13 *Let $n \in \mathbb{N}$. Then*

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(2n^2 - 1)$$