Math Camp **Summer 2018**

Lecture 0: August

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0.1 Mathematical Logic

In this section we study the fundamental elements for building the proofs. Before starting our journey, let me give you a motivation why we need more rigorous approach.

Example 1 *We always can write 0 = 1-1, by repeating it many times, we can write 0 =(1-1) + (1-1) + ... + (1-1). By taking first 1 out, we have 0= 1+ (1-1) +(1-1) + ... + (1-1). Therefore 0=1*

This elementary example shows that something is hidden behind "..." and in order to find out it, we need more mathematics then we used to learn in traditional Calculus course.

0.1.1 Logic and Statements

Logic is the framework upon which the rigorous proofs are built. Without some basic logical concepts, it would be impossible to structure proofs properly. When we prove theorems in mathematics, we are demonstrating the truth of certain statements.

Definition 0.1 *A statement is anything we can say, write, or otherwise express, which can be either true or false.*

Example 1(a) $x^2 = 9$

Example 1(b) *Equilibrium is Pareto Efficient*

Example 1(c) $f(x) = |x|$ *is continuous*

Example 1(d) *Alice is a PHD student*

Note 1 *For something to be a statement, it has to be either true or false in principle; that something is a statement does not depend on whether we personally can verify its truth or falsity.*

Remark 1 *We make two assumptions when deal with statements: every statement is either true or false, and no statement is both true and false.*

[0] All errors are my own.

0.1.2 Logical Operators

What makes statements more valuable for our purposes is that there are a number of useful ways of forming new statements out of old ones. In this section we will discuss five ways of forming new statements out of old ones, corresponding to English expressions: and; or; not; if, then; if and only if.

Definition 0.2 *Let P and Q be statements. We define the* ***conjunction*** *of P and Q , denoted $P \wedge Q$, to be the statement that true if both P and Q are true , and false otherwise.*

The best way to understand the definition of $P \wedge Q$ is by the "truth table"

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Remark 2 *The truth table shows whether the new statement is true or false for each possible combination of the truth or falsity of each of P and Q.*

Example 2 *Let $P =$"it is raining today." and let $Q =$"it is cold today".*

. The statement $P \wedge Q$ would formally be "it is raining today and it is cold today."

For our next definition, once again let P and Q be statements.

Definition 0.3 *We define the* ***disjunction*** *of P and Q , denited $P \vee Q$, to be the statement that true if either P is true or Q is true or both are true , and false otherwise.*

The truth table for $P \vee Q$ is given by

p	q	$p \wedge q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 3 *A simple example is a statement "In your first semester you will take ECON 629 or you will take STAT 630".*

The truth of this statement implies that at least one of the statements is true, The only thing not allowed is that both statements are false.

Remark 3 *In mathematical usage the statement "In your first semester you will take ECON 629 or you will take STAT 630." interpreted possibility of both " taking ECON 629 and STAT 630" statements being true as an entire statement "true" option. If I wanted to exclude this possibility, I need explicitly say "In your first semester you will take ECON 629 or you will take STAT 630, but not both."*

For our next definition, let P be a statement.

Definition 0.4 *We define the **negation** of P , denoted $\neg P$, to be the statement that true if P is false , and is false if P is true.*

The truth table for $\neg P$ is given by

p	$\neg p$
T	F
F	T

Example 4 *Let P= "Alice is a PHD student.*

Then the most straightforward way of negating this statement is to write $\neg P =$ "It is not the case that Alice is a Ph.D. student." or an easier way to say is "Alice is not a Ph.D. student".

The last two ways of combining statements are connected to the idea of logical implication. Usually, many of the students confuse the meaning of implication. Consider the statement *" If Alice studies hard, she will pass a qualifier exam."* What would it mean to say that this statement is true? It **would not** mean that Alice is studying hard, **nor** would it mean that Alice will pass a qualifier. **The truth of this statement** only means that if one thing happens (namely Alice studies hard). then another thing will happen (namely Alice will pass a qualifier exam). In other words, the one way in which this statement would be false would be if Alice studies hard, but does not pass a qualifier exam. The truth of this statement **would not** say anything about whether Alice will or will not study hard not would it say anything about what will happen if Alice does not study hard. In particular, if Alice did not study hard, then it would not contradict this statement if she passed a qualifier nonetheless. This is a very important point, that you need to remember for proofing something. You will appreciate the implication of this point when we will do real proofs. We formalize our approach as follows

Definition 0.5 *Let P and Q be statements. We define the **conditional** from P to Q , denoted $P \to Q$, to be the statement that true if it is never the case that P is true and Q is false.*

The illustration in the truth table is:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The first two rows of the truth table are intuitively quite reasonable . However, the third and fourth rows of the truth, which say that the statement $P \to Q$ is true whenever P are false, regardless of the value of Q, are less intuitively obvious. Remember that the one situation we are primarily concerned with is that we do not want P to be true and Q to be false.

Example 5 *Let consider a conditional statement "if it rains today, then I will see a movie this evening."*

Here P = 'it rains today," and Q ="I will see a movie this evening." Try to analyze this statement and understand what the truth of this statement says and does not say. We will talk about this example in class. For our final definition, let P and Q be statements.

Definition 0.6 *We define **biconditional** from P and Q , denoted $P \leftrightarrow Q$, to be the statement that is true if P and Q are both true or both false, and is false otherwise.*

p	q	$p \to q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 6 *An example of a bi-conditional statement is "I will teach this course if and only if (iff) I will get paid."*

This statement has the form $P \leftrightarrow Q$, where $P =$ "I will teach this course," and $Q =$"I will get paid". The truth of this statement does not say that I will teach this course, or that I will get paid. It says that either I will get paid and I will teach this course, or that neither of these things will happen. In other words, it could not be the case that I will get paid and yet I do not teach this course, and it is also could not be the case that I teach, and yet I do not get paid.

Let end this section by introducing two more important concepts.

Example 6(a) $P \vee \neg P$ *(**tautology**)*

Example 6(b) $P \wedge \neg P$ *(**contradiction**)*

Try to build the truth table for above two concepts and think about what they mean. We will talk about more about this concept during the class.

0.1.3 Relations Between Statements

In this section, we start to discuss relations between statements. Usually, relations between statements recognized as "meta-statement" that we make about statements. An example of a meta-statement is the observation "if the statements". We will study two examples, namely **implication** and **equivalence**, which are the meta-statement analogs of conditionals and bi-conditionals. Recall that the intuitive idea of logical implication is that statement P implies statement Q if necessarily Q is true whenever P are true. In other words, it can never be the case that P are true and Q is false. We start the study of the implication by example.

Example 7 *Show that $\neg(M \to N)$ implies $(M \vee N)$*

We show this implication in two ways. First, we can check the truth tables for each of $\neg(M \to N)$ and $(M \vee N)$.

The column numbered 4 in the first truth table has the truth values for $\neg(M \to N)$, and the column numbered 3 in the second truth table has the truth value for $(M \vee N)$. We observe that in any row that has T as the truth value for $\neg M \to N$, there is also a T for the truth value of $(M \vee N)$ (In our case there is only one such row, but it is not important). Note, it makes no difference what happens in the rows in which $\neg M \to N$has truth value F. Thus $\neg M \to N$ logically implies $M \vee N$. Alternativelt, rather than having two truth tables to compare, we can use **conditional** to recognize that our observations about the above truth tables is the same as sayiong that the single statement $\neg(M \to N) \to M \vee N$ will always be true. In other words, the statement is a **tautology**. This last statement we will illustrate in class.

M	N	$\neg$	$(M$	$\rightarrow$	$N)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	F	F	T	F
		4	1	3	2

M	N	M	$\vee$	$N)$
T	T	T	T	T
T	F	T	T	F
F	T	F	T	T
F	F	F	F	F
		1	3	2

Definition 0.7 *Let P and Q be statements. We say P* ***implies*** *Q , if the statement $P \rightarrow Q$ is a tautology. We denote it by $P \Rightarrow Q$.*

Note 2 *It is important to differentiate the notions "$P \Rightarrow Q$" and "$P \rightarrow Q$". The notion $P \rightarrow Q$ is a statement; it is built up out of the statements P and Q. The notion $P \Rightarrow Q$ is a meta-statement, which in this case is simply means that $P \rightarrow Q$ is not just true in some particular instances, but is a tautology.*

In Example 7 statement $P = M \rightarrow N$ and $Q = M \vee N$ and the implication $P \rightarrow Q$ is tautology.

Example 8 *Show that $(M \rightarrow N) \leftrightarrow (\neg M \rightarrow \neg N)$*

Definition 0.8 *Let P and Q be statements. We say P and Q are* ***equivalent*** *, if the statement $P \leftrightarrow Q$ is a tautology. We denote it by $P \Leftrightarrow Q$.*

Note 3 *It can be seen that $P \Leftrightarrow Q$ is true iff $P \rightarrow Q$ and $P \leftarrow Q$ are both true.*

0.1.4 Problems

Problem 1 *For statements P and Q, show that $(P \wedge (P \implies Q)) \implies Q$ is a tautology. Then state $(P \wedge (P \implies Q)) \implies Q$ in words. (The is an important logical argument form, called* **modus ponens.)**

Problem 2 *Show that $\neg(\neg P) \Leftrightarrow P$ (***Double Negation)**

Problem 3 *Show that $A \vee B \rightarrow Q$ equivalent $(A \rightarrow Q) \wedge (B \rightarrow Q)$*

Problem 4 *Show that $(P \leftrightarrow Q)$ equivalent $(P \rightarrow Q) \wedge (Q \rightarrow P)$*

Problem 5 *Show that $(P \rightarrow Q) \rightarrow (\neg P \rightarrow \neg Q)$*

Problem 6 *Show that $(\neg Q \rightarrow \neg P)$ equivalent to $(P \rightarrow Q)$ (**Contrapositive**)*

Problem 7 *Show that $\neg(P \rightarrow Q) \iff (P \wedge \neg Q)$ (**Contradiction**)*

0.2 Homework Assignments

Problems 2, 4, 6, and 7.

0.3 References

Ethan D.Bloch, 2000, *A First Course in Abstract Mathematics*
Sterling K. Berberian, 1994, *A First Course in Real Analysis*

0.4 Appendix

Solution for problem 1.

Solution 1 *A compound statement is a tautology if it is true for all possible combinations of truth values for its component statements. The truth table below shows that* $(P \wedge (P \implies Q)) \implies Q$ *is indeed a tautology.*

P	Q	$P \implies Q$	$P \wedge (P \implies Q)$	$P \wedge (P \implies Q) \implies Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

An example of $(P \wedge (P \implies Q)) \implies Q$ *in words is the statement:*

If P, and P implies Q, then Q.